**Path Integral Formulation**

Let’s now look at one more angle on solving the time-dependent Schrodinger equation. This approach was developed by Richard Feynman and is an important tool in quantum field theory, and other areas of physics. So once again consider the time-dependent Schrodinger equation.



A standard way to solve such equations, which arises in the study of differential equations and partial differential equations, is to look at an ancillary Schrodinger equation.

**Green’s function (a.k.a. the Propagator)**

This ancillary Schrodinger equation is given below. It is the same as the equation above, but with a different initial condition.



GP(x,t) is called the ‘green’s function’, and also the ‘propagator’ – for reasons we’ll understand shortly. To emphasize that GP depends on what x′ is as well, we often write GP(x,t|x′). Once we’d determined what this GP is, then we can determine what the answer to our original problem is, since we can say:



You can verify that this expression does satisfy the Schrodinger equation. For instance,



And you can also verify that it does satisfy the initial condition. Now, what is GP(x,t|x′)? Well generally, any position space function which satisfies the Schrodinger equation can be written as:



where |ψ0> is the initial state. Alright, well the initial state of GP is the position eigenstate at x′, i.e., |x′>. So we have:



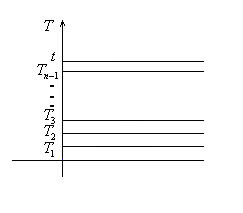
Physically speaking then, GP is the probability (amplitude) that a particle initially in the state |x′> will have evolved into the state |x> by time t. In other words, it is the probability amplitude that a particle initially at x′ will have traveled (propagated) to the point x in time t. This is why GP is also called the propagator.

**Break up t into N segments δt = t/N**

Now we will investigate how to calculate GP, and hence, ψ(x,t). First, remember the composition property of the time-evolution operator.



We will use this as follows. Break up t into N segments, each of width δt = t/N. Eventually we will take N to ∞ so δt is very very small.



And then we can write U(t) as a product of time evolution operators each propagating over a segment of the time of the total time interval.



Now insert this into our expression for GP, and insert the identity operator inbetween each U.



So what we have now is an n dimension integral over a product of n little Green’s functions, instead of just the one big Green’s function we had before. How does this help? Because we can evaluate these tiny propagators since the change in time is super small. We do so with the help of the formula for U from the Raleigh-Schrodinger time-dependent perturbation theory.

**Evaluate tiny propagators by neglecting time-ordering operator, reduce VI to V**

Consider that:



And remember that the time ordered exponential can be written as,



Now there is a simplification afforded by the fact that the interval of integration, [t, t + δt] is so small. The simplification is this: we can effectively ignore the time ordering prescription because the error in doing so is proportional to the integration interval and since it is very small, the error will be very small. For consider,



This merely illustrates that the error in commuting these two operators is of order δt, and if δt is very small, then we can neglect this error. Therefore, for small times δt we have:



Additionally we can approximate I(t) by simply (t) since δt is small. Because…



So consistent with our approximation above, since we will ultimately be taking the time interval δt to 0, we can replace I(t) with (t). So altogether we can say that,



Alright, with this information, let’s figure out what our tiny propagators are:



Again using the fact that tb – ta = δt is very small, we can write,



and so we can write our expression as:



So the quantity in the exponent you might recognize from classical mechanics. It is related to the so-called *action*. The action of a particle is the time integral of the Lagrangian over some path, where the Lagrangian is just the difference in the particle’s kinetic and potential energies. In other words,



where it is understood that x(ta) = xa and x(tb) = xb. Now in the limit that tb → ta, the integral can be approximated by the first term in its Riemann sum. And so we would have:



(making boundary conditions explicit) and so we can write:



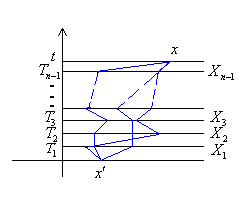
This is a very important identity and is the basis for the path integral formulation of quantum dynamics. Now we’re almost at our final result. Let’s plug in our formula into our expression for the Green’s function,

**Integrate over all little propagators → path integral**

We get (suppressing the t arguments in S for simplicity):



where in the third line we use the fact that all the time differences T3 – T2, etc., are just t/N. This looks really bad right now – as most things do in Quantum Mechanics (at first) – but it can be simplified into a very beautiful expression – as most things can in Quantum Mechanics (eventually). So step back a minute and consider what this expression means. Going right to left, we start at x′ at time T = 0, and traverse to X1 at time T1 and calculate the action in between. Then add to it the action going from X1 to X2 between times T1 and T2. Then we add to that the action going from X2 to X3 between times T2 and T3. We keep doing this all the way to x at time t. So the net value of the exponent is just the total action, along the specified path x′ → X1 → X2 → X3 → … → Xn-1­ → x, starting at x′ at T = 0 and ending at x at time T = t. But then that the integrals allow the intermediate points X1, X2, X3, …, Xn-1­­ to lie anywhere along the real line, and so integrating over all these intermediate points amounts to summing up the total action over all possible paths starting from x′ at T = 0 and ending at x at T = t. This is illustrated below as a heuristic diagram, indicating that we are to sum the action over all possible paths which originate at x′ and end at x.



So we can write Green’s function as:



where it is understood the path begins at x´ and ends at x. This sum over paths can be heuristically converted into an integral, just like a sum over points can be. Such an integral is called a path integral, and denoted below:



The integral D[X(T)] is over all possible paths that the particle may take from x′ to x between times 0 and t. When evaluating the path integral, often the most important contribution comes from the so-called classical path and so we can write,



where A(t) is some proportionality factor (and it is noted S depends on x and x´ because all paths must end and begin there respectively). The exact form of A(t), as depicted above, can be derived via the time-dependent WKB approximation. Two equivalent expressions are:



(the integral is w/r to t because S, evaluated at the classical path, depends on x, x´, and t (because path ends at x at time t) Could perhaps also derive it from the required initial condition GP(x,0|x´) = δ(x-x´). We’ll note that the classical approximation will be exact for cases where V(x,t) ~ xp, where p ≤ 2. This follows from the fact that such cases constitute a Gaussian functional integral, which can be done exactly.