**Perturbative Expansion of GF**

**Example. HO in constant external field**

Let’s take simple case of this guy – a harmonic oscillator in a constant external field:



Let’s consider the equation of motion of the x operator. It follows classical laws. So,



And solution is:



We can identify the annihilation/creation operators as usual, sort of:



where we use the subscript H to indicate these are the annihilation/creation operators of the interacting H; the ones without the subscript are the old (now interaction picture) creation/annihilation operators. And let’s verify they’re normalized,



And so we can write, using pH = mH:



Plugging these into H (can evaluate at t = 0 because time doesn’t matter), we have:



Okay so finally,



It just changes the zero-point energy, as we’d expect from classical mechanics. Let’s get some GF’s, expectations taken against the interacting GS. Which is:



where x(t) is in the *interaction* picture here. Can make it plausible that |GS> = S(0,-∞)|GS0> (where |GS0> = |0>). For instance, consider the following crude manipulations:



This is just the translation operator acting on |0>. Acting on |0>, or any state, it would shift it to the left by h/k. This is precisely the position of the new equilibrium point as (1/2)kx2 + hx has its minimum at x = -h/k. So that checks out. Another way to see this is through the equation for |GS>.



and projecting onto the position basis, we have:



Changing variables to u = x + h/k, we’d have:



This is the same equation as for the non-interacting GS. And so the solution would just be:



which coincides with the fact that the ‘interacting’ ground state is just the old one, shifted over by -h/k. OK, *now* let’s do the GF’s. We’ll take expectations against |GS>.

**Exact Greater GF against |GS>**

Let’s do the greater GF. Skipping steps using our non-interacting results, we have:



**Exact Lesser GF against |GS>**

Let’s do the lesser GF.



**Exact Causal GF against |GS>**

What about the causal GF:



Note the causal GF obeys the ODE:



where,



This isn’t exactly known a priori I’d say. But if we could intuit that <x> will just be the equilibrium point of the oscillator, which is where the forces balance, i.e., where -kx – h = 0 → xeq. = -h/k, then we might suppose this to be:



So then our equation would be:



Filling in our proposed solution, we’d have:



The implicit exponential convergence factor would also be there. So that checks out at least. Let’s look at some more GF’s. Let’s do the anti-causal GF.

**Exact Anti-causal GF against |GS>**

The anti-causal Green’s function can be obtained in similar fashion:



The implicit exponential convergence factor would also be there.

**Exact Retarded/Advanced GF against |GS>**

So the retarded is:



and because our xH(t) differs from the ‘free’ x(t) by only a constant, we can see that the retarded GF will be no different than it was before. Same will be true of the advanced GF. So,



The implicit exponential convergence factors would also be there.

**Perturbative expansion of causal GF against |GS>**

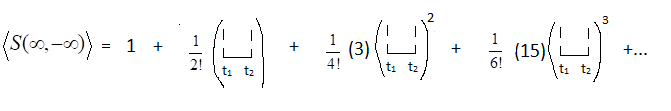
Now let’s consider the perturbative expansion of the causal GF. This would be:



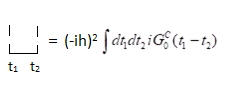
Consider first few terms of S:



Only even powered terms will survive in S. And it will be, counting multiplicities in constructing the diagram:

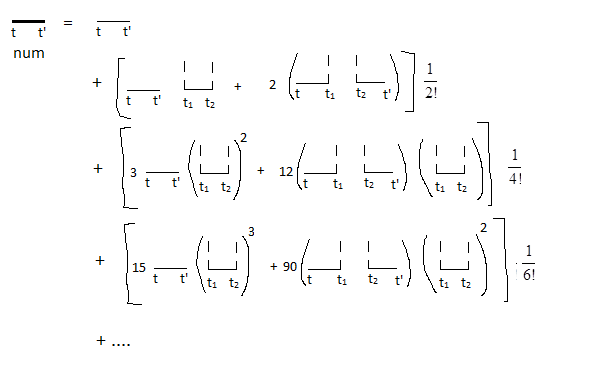


where,

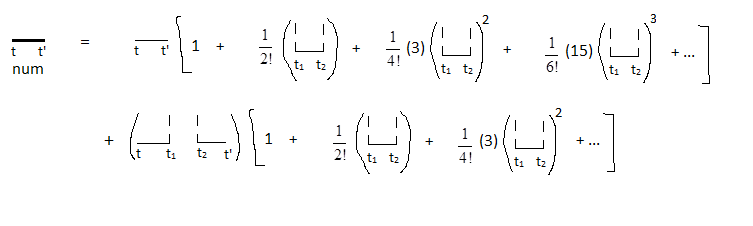


FWIW, should note that ½! = ½, which is the symmetry factor of the first diagram (t1 and t2 indistinguishable). And 3/4! = 1/8, which is the symmetry factor of the second diagram (b/c t1t2 are indistinghishable, t3t4 are indistinguishable, and {t1t2} {t3t4} are indistinghishable). And finally…15/6! = 1/48 = (½)3(1/3!) (b/c t1t2 are indistinguishable, t3t4 are indistinghuishable, t5t6 are indistinguishable, and the sets {t1t2}, {t3t4}, {t5t6} are indistinguishable.

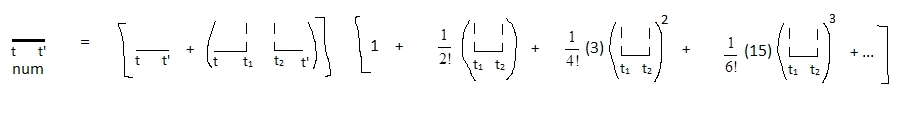
And the (numerator of the) GF itself will be (again noting that only even powered terms in S will survive the contraction with x(tα) and x(tβ)):



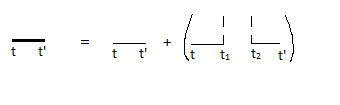
Let’s note that we can rewrite this as:



which is, we suspect,



And so we see the vacuum bubbles do cancel out. And so we have:



The left term in the parentheses is, making the implicit exponential factor explicit:



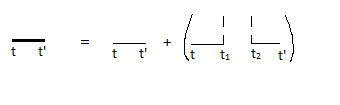
and the right term would be:



So then see that:



So this checks out! It would’ve been easier to do this directly from Fourier space, as it turns out. Going back to:



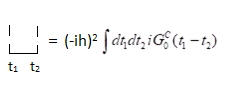
we’d have:



and so,



That was easier. One last thing. We’ll note that the terms in S, the vacuum bubbles themselves, are not finite. For instance,



is infinity as we can see from our evaluation of just half that diagram a little above. So good thing it cancels. One more last thing. Let’s consider,



Now let’s consider the variance of the position about its equilibrium point xeq = h/k. Well, this is:



And this is just the variance of the unperturbed oscillator. This makes sense because the only difference between the perturbed and unperturbed oscillator is the equilibrium displacement.

**Exact Causal GF against |0>**

Now let’s consider the interacting causal GF against the free-particle GS = |0>. This would be:



where both tα,β are on the outgoing part of the contour. First let’s evaluate this from the top line. We’ll have to put the operators in terms of the old creation/annihilation operators to evaluate the expectation against |0> …



And then,



and for what it’s worth, it is clear that if we started time development at t0, instead of t = 0, we would’ve gotten (note |0> is the state we start with at time t0):



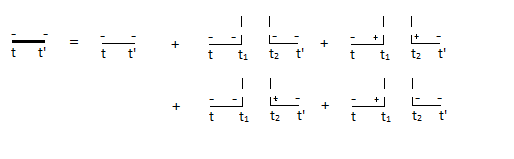
Might observe that the |GS0> GF oscillates about the |GS> GF. This kind of makes sense too, because if we have a spring with a mass on it, and slowly turn on the gravitational field, it will settle at the equilibrium xeq. = -h/k. But if suddenly turn it on, it will just perpetually oscillate *about* it. That’s what we find here.

**Perturbative Expansion of Causal GF against |0>**

And now the diagrammatic expansion, from:



We know vacuum bubbles cancel. So we’ll have:



And this is all we’d have to *any* order. Mathematically, this is:



There are no ½ factors on the - - and ++ vertex guys because the external legs make them not interchangeable. Putting everything in terms of >, <, we get:



This must be independent of τ. So let’s see how that is true. The terms grouped together ought to do the job.



So then,



and further,



Well that’s good. Now let’s fill in these functions,



Continuing,



And so,



Just as we found above. This is not a function of t – t´? No, because |0> is not eigenstate of H. And if we started off in |0> at t0 rather, we would’ve gotten:



So everything checks out. And let’s look at the variance of the position about its equilibrium point. This is:



This is obviously not time-independent, and not what we got when we evaluated this quantity for GC(t,t´) taken against the |GS>. This coheres with the fact that turning the perturbation on when the particle is in the ground state of the free system, |GS0>, makes it oscillate about the perturbed ground state |GS>. But turning the perturbation on when the particle is in the true ground state, |GS>, does nothing to it.

**Example. HO in time-dependent external field**

Now let’s say:



Turns out we can solve this situation exactly too. x’s equation of motion would be:



We have:



where G0R is the solution to the equation:



Of course this function, as the notation suggests, *does* just happen to be our unperturbed retarded GF, which is:



Let’s check that we get the correct result when h(t) is constant,



This does match our previous result. So in general, then, we have:



I don’t think we can identify creation/annihilation operators for the overall H, since H is not constant. So let’s just focus on (causal) GF expected against the free-particle ground state |0>. And we’ll put xH(t) in terms of the free annihilation/creation operators:



**Exact Causal GF against |0>**

From our equation for xH(t), we would have:



It is fairly evident that if we started development at t = t0 (i.e. if our initial state were |0> at t0), this would generalize to:

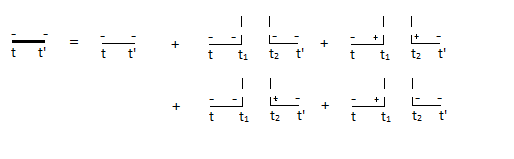


**Perturbative Expansion of Causal GF against |0>**

Now let’s work it out from the perturbative expansion:



(where tα, tβ would both be on the ‘outgoing’ part of the contour). We know the vacuum bubbles cancel out. So we’d just have, exactly,



Reviewing our calculation for constant h, we can see this will come to:



Now let’s fill in these functions,



And so,



which also matches. Cool. And we can get the expected things from this. For instance,



and,



**Example. Harmonic Oscillator with anharmonic term**

Consider,



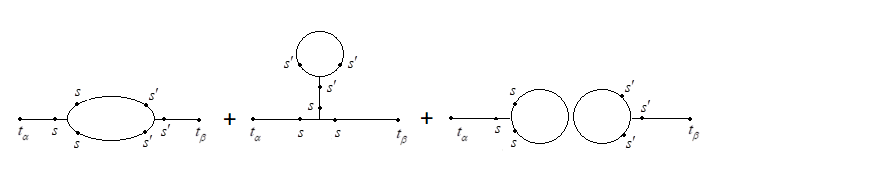
And say we want GC(tα,tβ), which is:



Then this is given by, (recalling |GS0> is denoted |0> too):



Let’s work out the first two terms in the expansion. The first is just the bare GF. The second is zero because we have an odd number of terms – and that means a bunch of odd numbers of a’s or a†’s, which must ultimately annihilate the vacuum. So the only term left is the squared vertex. These should be the terms that survive.



We’ll note that the first term could be constructed in many equivalent ways. Options are:

1. connect α to unprimed or primed vertex (2)

2. connect α to leg 1, 2, or 3 (say 1 for sake of discussion) (3)

3. connect 2 to 1´, 2´, or 3´ (say 2´ for sake of discussion) (3)

4. connect 3 to 1´ or 3´ (say 3´ for sake of discussion) (2)

5. connect remaining leg to β (1).

So there are 2×3×3×2×1 = 36 possibilities here. And there is a factor of (1/2)(1/3!)2 coming from V itself. So this diagram gets a net factor of 1/2. Then for the middle diagram…

1. connect α to unprimed or primed vertex (2)

2. connect α to leg 1, 2, or 3 (say 1 for sake of discussion) (3)

3. connect 2 to 1´, 2´, or 3´ (say 2´ for sake of discussion) (3)

4. connect 3 to 1´ or 3´ (say 3´ for sake of discussion) (2)

5. connect remaining leg to β (1).

So the multiplicity is 36 again, and that gives us a net factor of ½ again. The last diagram’s multiplicity would work out like this:

1. connect α to unprimed or primed vertex (2)

2. connect α to 1, 2, or 3 (say 1 for sake of discussion) (3)

3. connect 2 and 3 together (1)

4. connect β to 1´,2´, or 3´ (say 1´ for sake) (3)

5. connect 2´ and 3´ together.

So there are 18 possibilities in the last. And it would have an overall factor of ¼. So we have:



And we can work out the factors from the Feynman rules instead. The first diagram has two indistinguishable propagators → ½! factor. Note the vertices are apparently not indistinghishable, I guess because they are connected to those external points, which serves to ‘distinguish’ them. The second diagram has a coincident propagator → ½. And the third diagram has 2 coincident propagators → (1/2)2.