**RSTDPT (Constant)**

**Constant (in time) Perturbation**

Now let’s presume time-independent V’s.



Note this problem is theoretically the same as switching the perturbation V(x) on at time t = t0. It doesn’t really matter if we switch it on at t0, or if it’s always been on up to that point, since we are specifying the initial condition at t = t0 to be ψ = ψ0, the subsequent dynamics will be the same either way. Our situation could be exactly solved by finding the eigenbasis of the interacting problem, but we’ll see how far we can get with Perturbation Theory. Consider the GF, defined in a previous file,



and the perturbative series it’s given by:



where,



and we have summation over repeated indices.

**G out to 2nd order**

Let’s examine the series out to 2nd order. Well, our zeroth order term is:



So that’s easy. So our zeroth order term is:



Our first order term is:



So we have:



Implicitly, we presumed En ≠ Ei. Then the En = Ei term can be obtained most easily by going back to the integral,



Or we can just take the limit that Ei --> En. Now let’s take a look at the 2nd order term,



Proceeding,



which is:



Now gotta do that integral,



and we come to:



Going to pretty it up,



Gonna split the first term into b = n, b ≠ n. And going to eliminate all the δn≠I expressions, as we’re implicitly assuming n ≠ i. So then,



Then can combine terms,



which is:



So then finally, we come to:



(implicit summation over b index) And that’s why we don’t want to go out to 2nd order. What about when En = Ei? Let’s go back to the integral in this case (or we can take the limit Ei -> En),



Simplifying a little,



Well, this is G, and technically, we’d want S, rather. But all we have to do is take off the Gnn(0)(t-t0) prefactor from all the terms. So putting this all together, we’d have (n ≠ i, and implicit summation over b index):



As usual, although we already have it, we could find the Snn term by taking the limit Ei -> En.

*Conditions on Convergence*

In general, for perturbation theory to be valid, we usually need successive terms to be smaller than the previous ones. Looking at the first order term (in the green highlighted box), it seems we need: |Vni| << |En – Ei|. This was the same requirement as we had for regular TIPT. In the second order term, it looks like we need both |Vab| << |Ea – Eb| and |En – Ei| << 1/t. So generally,



*General Behavior*

Let’s look at S in different limits. What is the small t – t0 limit?



So we have:



How about t ~ 1/|En – Ei|? Further, let’s say that |En – Ei| is small compared to the majority of energy level spacings, but not smaller than |Vni|, say, to be consistent with our highlighted inequality. Then we can say,



where we keep only the first term, because it should be larger than the first crossed out term, due to the denominator, and larger than the other two crossed out terms, due to the fact that there’s N-2 (N = number of energy levels) of the kept term, and only 1 each of the latter two crossed out terms. Here we also use the fact that (En – Ei)2 is not presumably so small as to be singular (b/c of our inequality |En – Ei| > |Vni|), which would otherwise make the last two crossed out terms quite large. So we can say,



We can write this more succinctly by defining the T-matrix (implicit summation over the b index, well, except b = i):



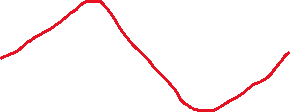
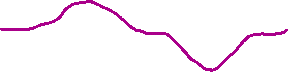
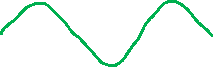
Then we have:



So I think we can glean the following from our results. In the small t limit, looks like the states will transition |i> 🡪 |n> with a probability proportional to Vni(t-t0). If this direct transition matrix element is zero, then indirect transitions are possible, but these are less likely for small times, going as ~ ΣbVnbVbi(t-t0)2 or something. When time is at its max for our range of applicability, say t ~ 1/|En – Ei|, then these are of order Vni/|En – Ei|, or ΣbVnbVbi/|En – Ei|2. So we can see that the transition amplitude is greatest when |i> and |n> are close together energy-wise, all things considered. After this, the transition amplitude generally drops back down and oscillates, with period 2π/|En – Ei| as the formulas suggest, and the few explicitly solved cases (spin s = ½, s = 1) demonstrate. The general form of Sni looks like this roughly,



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The blue curve would be Sni(t) if |i> and |n> are relatively far apart in energy (but Vni relatively large), the green curve is if they are closer in energy (but Vni smaller), and the red curve is if they are closest in energy (but Vni smaller still, for no particular reason). Remember the initial slope of the curves is proportional to Vni and the amplitude of the curves is something like Vni/|En – Ei|. The purple curve is something like what we’d get if Sni(t) is zero to first order in V and so goes as a quadratic for small t. There is nothing saying that any one of those curves ought have the highest peak. It depends on what Vni/|En – Ei|, or ΣbVnbVbi/|En – Ei|2 is. Should note that as Ei –> En our discussion would seem to imply that the transition amplitude will grow without bound, and also oscillate with an increasingly large, tending towards infinite, period. But note that we will have then violated our highlighted inequality, since we’d have |En – Ei| < |Vni| in this scenario. Basically, what seems to happen *then* is that we can only take t out to t ~ 1/|Vni|. So the transition will max out at ~ Vni/|Vni|, or ~ ΣbVnbVbi/|Vni|2. And it will oscillate with period around 2π/|Vni|. Physically, this comes about from perturbative corrections to the energy, which fundamentally is what governs the frequency of the time-development.

**Perturbative Transition Rates**

Let’s calculate the transition rate,



to lowest order in PT. This will be:



And so we have:



Now let’s get it to next highest order? Well, we’ll move to the limit that En ~ Ei again, and consider the Δt ~ 1/|En – Ei| limit. Then the same analysis will show that, to next order at least,



This formula says that the probability of transition from Ei --> En basically grows as 2(t-t0)|Tni|2 (qualitatively true at least) until t – t0 ~ 1/(En – Ei), at which point it will begin to descend back down and oscillate back and forth with period ~ 2π/|En – Ei|. Again, this would seem to imply an infinite scattering rate in the long time limit, and a correspondingly infinite period of oscillation, as Ei – En -> 0. But we cannot conclude this from our perturbative formula because such small energies would violate the inequality |Vni| < |En – Ei| < 1/t. Rather, in this limit, it looks like it will grow as 2(t-t0)|Tni|2 until t – t0 ~ 1/|Vni| or so, and then oscillate back and forth with a period of ~ 2π/|Vni|. Physically, this comes about from perturbative corrections to the energy. We’ll see this again in the next file as being the dominant contribution to the scattering rate in the long time limit, especially when the two states |n> and |i> are scattering states, i.e., states in the continuum, which can be arbitrarily close to each other.

**Appendix A: GF from Fourier space**

Since V(t) is time-independent, it would be formally advantageous to take the temporal FT. We’ll do this and reproduce all of our results above. What is the FT of G(0),



Doesn’t technically exist. But, since we’ll be ultimately interested in G(t), we can use for G(ω) the function whose inverse transform gives us G(t). This is:



Since (using residues stuff),



[another way to justify the exponential convergence factors it to multiply both sides of the expansion equation by eε(t-t0) and then for each V(t´) add in eεt´ e-εt´. Then all G0’s will be of form G0(t-t´)e-ε(t-t´), which does have a FT, i.e., the one we’re working with above. Then we can take the ε → 0 limit]. Okay well then taking the FT of both sides of our expansion, using our Fourier transform results file, we have:



where we’re implicitly summing over repeated indices. And our series will explicitly read:



In terms of the operator GF, we have:



Note we can sum the geometric series here, and conclude that in fact the total GF is, at least formally,



the exact Green’s function.

**Examining first couple terms**

Let’s examine a few of these terms…



Now recalling,



we can say, presuming t > t0,



The zeroth order term is clearly,



The first order term would be…I’ll use residue stuff…



And so we can in fact write this as:



This matches our result above. When doing the residue thing, I kind of implicitly assumed n ≠ i, but we can get the n = i case by being careful about the limit n 🡪 i. We’d have:



Working on the second order term we have:



Working this out, we get (remember implicit summation over α):



(remember i is implicitly not equal to n here) Can simplify a bit,



We can split the bare term in the top line into:



And so we have:



This does match our previous result. Again, implicitly presumed n≠i. The n=i term would be:



Everything matches what we found previously.

**Appendix B: Simpler Form for Sni(t,t0)**

Going back to Sni(t,t0), and again remember there is implicit summation over b.



Just wanted to show that in Sni(2)(t,t0), we can get the b=i, b=n terms in the bracket by taking the b≠i, b≠n term at the very top, and taking the limit that b🡪i, b🡪n. I’ll show how this can be done for b🡪i.



So there. And presumably a similar analysis would show this to work for the b->n limit, or n = I for that mater. Given that, we could write S *much* more succinctly as (remember implicit summation over b index),



if we’re just careful to take the appropriate limit when evaluating those b=i, b=n, n=i cases in the first and second order parts of it. What if we try to normalize this?

**Appendix C: Normalizing S**

S obviously isn’t normalized. What if we try to find some N such that:



Then we have:



(implicit summation over b, including b = n) And really, it’s √N we want,



**Appendix D: A Different Perspective**

Let’s do the perturbative expansion this way. I want to basically see that all the t-terms come from the perturbative expansion of the energy. Start with our H:



And we’ll calculate G by expanding it in a basis of exact eigenstates. Going to use a little bit different notation. En is now the unperturbed energy, while n is the perturbed/exact energy. Likewise |ψn> are the unperturbed states, and |n> are the perturbed/exact states.



Now we also have the expansions, from RS Time-Independent Perturbation Theory,



Filling in just the first order terms, we have:



which is:



We can see that this matches our result at the top of the file, for G to first order in V. This makes it clearer where the terms in G’s perturbative expansion that are proportional to t come from. They comes from the V expansion of the n in the exp(-int) factor, where n, again, is the exact energy. Of course, we should be able to do the entire expansion this way. Let’s suppose that we use BW perturbation theory instead, expanding the wavefunction, but not the energy.



Filling this into our unperturbed G, to first order, we have:



And so we come to:



Hmmm.