**Perturbation Theory in 2 dimensions**

Didn’t think I would, but I do…want to work this out in 2D. I try to use 3D as guide and skip as much as possible.

**WKB Perturbation Theory**

So start with:



where V(r) is just some cylindrically symmetric potential. In cylindrical coordinates this simplifies to:



Now Lz2 is basically that ∂2/∂φ2 guy, and it commutes with the radial part of the kinetic energy operator. So we can write our wavefunction in this form.



(even if we didn’t recognize Lz and pz in the Hamiltonian, using the method of separation of variables would’ve resulted in the same trial wavefunction). Alright, now plugging this into our equation we get:



Now we recall that we had the issue where r is defined only over (0,∞), which doesn’t work for us in the WKB approximation. We need to put this equation in terms of a variable which goes from (-∞,∞). So we make change of variables:



So using chain rule, we have:



and so,



(could have written R(x) as R(ex) technically, like did for V, but whatever…point is we’re expressing it in terms of x) Fortuitously, the R´ term goes away. So we have:



and this equation is of the requisite form. OK, now we’re ready to apply the WKB approximation formalism to the problem. So let’s assume a solution of the form,



Filling this into our equation we have:



As before we will take the semi-classical approximation where the phase of the wave is assumed to be a smoothly varying function so that S′′ is small. Appending a λ to this term therefore, to prepare for a perturbation solution of this equation we get:



And now expand S in a power series in λ.



and fill into the S-differential equation,



The zeroth and first order equations are:



Solving the zeroth order equation we have:



and the first order equation solution is:



Then we can write our WKB approximate y(x) as:



and then, x = lnr, which implies:



Continuing to simplify…



Defining,



we can write our radial wavefunction as:



which is kind of what we might have expected, knowing the 1D and 3D results. Now we’re still not technically done. We have to determine what c1 and c2 are, etc. But at this point we can just use the analysis done before in the 1D case. The value of these coefficients depends on whether the effective potential here:



has both hard walls, one hard wall/one soft wall, or both soft walls. And the formula for the coefficients and energies will be identical to those determined before, in the 1D case. We won’t focus too much on the wavefunctions themselves, but for the energy levels we’ll have the formulas, just as before:



where A1 and A2 are the turning points of the effective potential.

**Application to the particle in magnetic field (symmetric gauge)**

Let’s apply this to a, well, particle in a magnetic field described by the symmetric gauge. So recall we had,



where **A** was the vector potential **A** = -(1/2)**r**×**B**, for a magnetic field **B** = B. V(r) is just 0. Working all that out, we get:



The equation can be simplified by going to cylindrical coordinates, as one might guess due to the presence of the x, y terms:



Then the expression simplifies to:



Filling in our typical ansatz,



this comes to:



and this now matches our archetypal radial form we analyzed above:



Our ‘potential’ is:



So the quantization condition is, since we have soft walls at the turning points:



and,



Let’s work this out. But we’ll clean it up first. We know from before that E will come out proportional to ℏωc. So let’s say E = ℏωcε. And let’s also put in terms of r in terms of the magnetic length: ℓB = √(ℏ/|e|B) so that r = ℓBξ. Then,



and so,



So our quantization integral is:



So our quantization integral is now,



That’s much nicer. Let’s now say ξ = √z. And let’s call ε´ = ε + (mℓ/2)sgn(e).

Then,



Now just need to get the roots of our guy, i.e., the turning points,



So we can say,



Now this is of the form of that integral we had to do in the 3D WKB file.



So we can say,



Working on the LHS,



So then,



and finally,



which is indeed what we found when we did this *exactly* in the B field Symmetric gauge file (given that our η ranges over η = 1,2,3,…,∞). Pretty cool as we should not in general expect exact results from this method! Now maybe we’ll look at the wavefunction,



Borrowing from 1D file, since we have soft wall turning points, we’d expect this to come out to:



Well let’s see. We need the turning points. These are at:



Let’s specialize to the ground state wavefunction, which has just one peak. That’s η = 1 (in our notation) and mℓ = whatever. So then,



So at least we can say that the wavefunction is peaked at around the average of these two:



Back in the file where we did this exactly, we found,



which, upon setting η = 0 (cause that file’s η = 0 corresponds to this file’s η = 1), matches our result. So actually the exact result is equal to our asymptotic result.