**Formal Properties of Green’s Functions**

Here are the 6 common green’s functions. For the Heisenberg picture operators, I think we must use the typical U(t,t0) = U0(t,t0)S(t,t0), as I don’t think the mixed phase convention time-development operator is compatible with the adiabatic theorem, which will be used later (see QM/Time-Dependent/Adiabatic Perturbation for discussion of phase convention). The time-ordering operator comes with a -ε, where ε = ±, for transposing the operators if they’re fermions/bosons, i.e, T{A(t)B(t)} = A(t)B(t)θ(t-t´) – εB(t)A(t)θ(t´-t). And the ± on the commutator means anticommutator/commutator. Note that the t argument is always with the A, and t′ always with the B. And the lesser/advanced GF are the only ones with the sign change.



These GF’s also implicitly carry with them exponential convergence factors that come in with the θ functions: θ(t) → θ(t)e-ηt, θ(-t) → θ(-t)eηt, where η = 0+ is an infinitesimal positive number, eventually taken to zero. Also, I think even the >,< GF’s carry these factors, and should be technically defined as, for instance,

G>(t,t´) = -iθ(t-t´)<AH(t)BH(t´)> - iθ(t´-t)<AH(t)BH(t´)>

→

G>(t,t´)= -iθ(t-t´)e-η(t-t´)<AH(t)BH(t´)> - iθ(t´-t)eη(t-t´)<AH(t)BH(t´)>.

Anyway, since this limit gives us 1 (unless t = ∞, or t´ = -∞), we will just leave it off, to save space, unless necessary. The exponential convergence factors are there, I think, because of the use of the adiabatic theorem:|GS> = Sη(t0,-∞)|GS0>, where the η is there to indicate exponentially slowly turning on the perturbation (see Quantum Mechanics/Time-Development/Adiabatic Perturbation). This is where eηt factors would show up in calculations. And I believe this convergence factor, initially in S, gets transferred to the GF’s and so effectively becomes part of their definition for any calculation involving times in infinite past or future (where there is a e-ηt term in the adiabatic theorem). For finite times, the exponential factors are superfluous, as their limit will always gives us 1.

In order for the Wick expansion, that we’ll use later, to hold, we must have that the background expectation state be an eigenstate of some bilinear H0 (or combination of such) |Ω0>



|Ω0> must also be ‘homogeneous’ – it cannot be a macroscopic quantum state, like a Bose condensate, which would seem to mean that the quantum occupation numbers must all be ~ 1/N. Furthermore this is only valid in the large V limit I think. The particle vacuum |0> satisfies these requirements, as does the Fermi ground state |F>. The operators AH(t), BH(t) can be anything, and may evolve in time according to some Hamiltonian (Schrodinger picture),



where both the single particle and two particle interaction may be time-dependent, though the latter is never (?) realized.

**Algebraic relationships between the various Green’s functions**

The green’s functions aren’t independent. Some algebraic relations between the green’s functions are as follows.



So we have,



And consider:



So we have:



Also consider,



and so we have,



and from the top relationship,



and so we have,



And we can also say:



**Symmetries of GF’s**

Consider a GF evaluated against some state |Ω0>. And presume time-development against a constant H. Constant H would mean energy is conserved. Then for instance our GF should depend only on t-t´, i.e., be time-translationally invariant.



Now if |Ω0> is an eigenstate of H, then:



But note that if we evaluated against states that weren’t eigenstates of H (like ones that were eigenstates of H0, rather), then this would not be time-translationally invariant.

**Green’s functions and the spectral function**

The following discussion will assume that we’re using the expectation basis, |Ω0>, which is an arbitrary state. We assume the operators develop according to a **time-independent** H for the spectral properties to hold. Now let’s look at the Fourier transform of the various green’s functions.

**Retarded Green’s Function**

Consider that we have the retarded Green’s function GR(t-tʹ). We can take the Fourier transform (including that implicit exponential factor aforementioned):



And finally this can be written as:



where we define a spectral function A(x), unfortunately the same letter as one of our operators. Note the spectral function is real.



Let’s pause to observe that the poles of G are the *many-body excitations*. And we’ll further note that if A and B are single particle operators, say ψ, ψ†, then the only non-zero overlap will be between states with a difference of 1 in particle number. And in this case only the single particle excitations therefore will be sampled. And if A and B are ρ, ρ, then overlap will be non-zero only between states with the same # of particles. So this will measure excitations obtained by basically swishing the particles around, rather than adding particles.

**Advanced Green’s function**

If we carry out the same procedure on the advanced Green’s function, we would evaluate the Fourier transform, and obtain:



**Greater Green’s function**

Now let’s consider the Fourier transform of the greater Green’s function,



The first two terms will give us the spectral function, and the second can be written as f(ω)A(ω), where we define f(ω) via the equality.



**Lesser Green’s function**

And we also have for the lesser GF.



We can put this in terms of the spectral function as well, and we’ll find:



Lesser GF has less terms, ha!

**Causal Green’s function**

And we also have for the causal GF.



We can put this in terms of the spectral function as well, and we’ll find, most simply from the algebraic relations at the top of the page



**Anti-causal Green’s function**

And we also have for the causal GF.



We can put this in terms of the spectral function as well, and we’ll find, most simply from the algebraic relations at the top of the page



**Properties of the Spectral Function**

We can see that the spectral function can be inferred by:



If A(ω) is real, then GA = (GR)\*, and we can say:



Moreover, we saw the Spectral Function was explicitly given by:



So it clearly has poles (infinities) at the excitations of the system (since |m>, |n> are the eigenstates, and so En – Em would be the excitations). But these are modulated by the overlap of the excitation states with |Ω0> and the operators A and B. The reality of A(x) seems to hinge on a lot of things. So if A and B are Hermitian conjugates of each other (or themselves), and if |Ω0> is an eigenstate of H, then A(x) should be real since:



And then its complex conjugate would be:



So there.

**Kramers - Kronig relations obeyed by GF’s and A(ω)**

Repeating the arguments from the single particle file, we have, *presuming* A(ω) is real:



where the Hilbert transform is:



and for the spectral function, we have:



and concommitantly:



**Expectations from Green’s Functions**

I suppose it is here where Green’s functions demonstrate their utility. The expectation of any single particle operator



can be calculated from its causal Green’s function



So say we start in some state |Ω0>, and it develops in time according to U(t), and we want to see how the expectation of A changes. Then we’d form,



(remember there is -ε cost for any transposition) For example, we can calculate the particle density:



with implicit summation over spin indices. We can get the spin density:



Now let’s say we have a general spin-independent Hamiltonian,



We can calculate the Kinetic Energy



So we have,



where the gradient acts on r alone. The single particle potential energy would be similarly,



(though the r+ is unnecessary) And we can calculate the interaction potential energy as well. Using (from 2nd quantized operators in varius bases),



we have:



so our two results are:



and the total energy would be the average of these two:



With the total energy and particle number available from calculation of the green’s function we can obtain the thermodynamic potentials, and hence calculate all quantities of interest.

**Expectations from the spectral function**

From the definition of G<, we have:



We also have,



This property actually follows from the definition of A in terms of GR and GA. So it is completely general. Let’s consider calculating <AB>. We can clearly find the appropriate linear combination of the two above.



And then we’ll have,

