**Tensor operators**

Now we’ll look at tensor operators.

**Cartesian Tensor operators**

When looking at how the operators **r**, **p**, and **L** transformed under rotations, we observed that they transformed as vectors do, i.e., D†(**α**)**r**D(**α**) = **r**′, where = exp{-i·**α**/ℏ}. In general, if we have set of operators Ti that transform in the same way:



then the set forms a ‘tensor’. And they are called tensor operators. Examples of tensor operators are **r**, **p**, **L**, and **S**. Written in terms of components this is:



(implicit summation over the j index) And it is said that when rotated Ti induces a representation of the rotation matrix. We can obtain another equivalent condition by taking this equation out to first order in **α**. Recall we had:



and taking it out to order δα, we have:



The first term can be worked out to give the identity matrix again. So we have:



We can write this as:



which we can write in shorthand as (Einstein summation notation):



Okay. So now putting this into our equation, and keeping only O(α) terms we have:



where in the last we change the dummy indices from m to k in the first term. Then we have:



and changing k → i, i → j, j → k we have a nicer looking equation:



and then we can cyclically permute the indices on the ε pseudo-tensor for free, and we arrive at:



**Spherical Tensor operators**

We can define a spherical tensor operator as a set of operators which transform under rotations by inducing a spherical tensor representation of the rotation matrix. The two equivalent conditions are:



(implicit summation over m’ index from -j to j), where,



and,



We can find an equivalent condition by doing the same thing we did above, and expanding the rotation out to first order.



Now let’s write as:



where u± are linearly independent (complex) vectors. Then we can say,



and since this equation must hold for all values of the independent components α+, α-, αz, we must have three separate equalities:



and so in general,



where we’ve written the √ thing a little differently, but equivalently.

**Example**

Consider a scalar operator, like = ·. This will transform like a rank zero spherical tensor operator T00. This is because:



and,



So transformation laws are same.

**Example**

Let’s show that one such rank 1 spherical tensor operator is , where Yℓm is a spherical harmonic but all of the position variables have been replaced by position operators.

So we’ll show that:



So first note that [using |**s**> to represent position ket too]



So the rotation operator rotates the argument of Y, as it would for any function as this wasn’t specific to Y. And now consider, taking the operator hat off of **r** for now.



Now this is a formal relationship, and so we can put the operator hat back onto **r**. And say,



and so altogether then:



So it’s proved.

**Example**

Let’s test the special case of **α** = (π/2)**k** on Y1m. So we have:



Performing the rotations we have:



whereas,



and,



So it checks out.

**Matrix elements of Spherical Tensor Operators**

The expectations of such tensor operators w/r to angular momentum like states follows general rules. First consider a state |njm> = |nj>|jm> which splits into a radial and angular/spin part. And then let’s examine:



We’ll start with one of the commutation relations we have:



which tells us that:



So one ‘selection’ rule is:



Another one is the Wigner-Eckart Theorem. This allows us to break the expectation into a magnitude and direction expectation product. Not bothering with proof.



The missing lower subscript on is to indicate that the result is independent of this subscript. The expectation is a Clebsch-Gordon coefficient, of the form <ℓ1ℓ2m1m2|ℓ1ℓ2ℓm>

**Example**

Some applications are the following. Suppose that we have a scalar operator T00. For instance, **x**∙**p**, or **x**·**x**, or some such. Then we’d have the rules,



and,



This means that j1 can range between j2 – 0 and j2 + 0. So j2 must equal j1. So we have selection rules:



**Example**

But suppose that we have a vector operator, . Then plugging this in we get selection rules,



But now m can be only 1, 0, or -1. So at least we can say that the Δm must be one of these values. And,



Now j1 can range from j2 – 1 to j2 + 1…so basically either j2 – 1, j2, or j2 + 1. So the difference j1 – j2 can be either 1, 0, or ,-1. So altogether we have the selection rules:



Note that Y1m(**r**) is a rank one spherical tensor operator of course. z itself could be written as Y10. But **r** can also be considered a spherical tensor operator T1q, because we can write a linear combination of the Y­1m(**r**) to write **r** = aY1-1(**r**) + bY10(**r**) + cY11(**r**). And since a linear combination of tensors is a tensor itself, we can legitemately represent **r** as T1q.

**Example**

Evaluate the non-zero’ness of :



The Clebsch-Gordon coefficient requires that: q + m′ = m, which implies that: Δm = 0, -1, +1. And Δℓ = 0, -1, +1 as stated above. This matrix element is often seen in the context of shining light on an atom. In that context, the electron would transition to other states after absorbing a photon, which has angular momentum ℓphoton = 1. In that sense, noting the angular momentum addition rules for two particles (see Many Body folder), we’d expect ℓ = ℓ′+1, ℓ′, ℓ′ - 1, which is consistent with our Tensor rule above.