**Translation Operator**

Now we’ll examine some continuous symmetry operators. Consider again the translation operator, generalized to 3D. Let’s determine the representation of this operator. Most of the work has already been done for us – you will recall that we found, when discussing the Heisenberg picture, that:



So it is easy to generalize this to 3D:



So we have the infinitesimal form of the translation operator, and we immediately recognize the momentum operator as the generator of translation. Given that **p** is the generator of translation, we can write the translation operator as:



And we have the following symmetry/conservation law:



Note that since we typically assume that momentum is conserved throughout the entire universe, this implies that the Hamiltonian of the universe is invariant with respect to spatial translations. Now let’s consider its effect on various other kets and operators. We will find:



The first is obvious, based on our work in the Heisenberg file. Next the momentum eigenket:



which is just |**p**> with a different prefactor. So translation doesn’t change the momentum eigenstate’s eigenvalue, but it does change its phase. And on an angular momentum eigenstate:



And since the argument of Yℓm only matters in so far as the angle specified, we’d have to interpret the argument **r**-**d** as referring to its angular coordinates, while |**r**-**d**| would not matter. As for spin states….of course,



Since the translation operator doesn’t act on the spin part of the HS. Generally speaking the action of the translation operator on a wavefunction, in position space is:



Observe how this agrees with our expression for |ℓm>. Now let’s look at the effects on operators. We’ll have:



Starting at the top…



Working out the first commutator we have:



This not being an operator, this makes all succeeding commutators 0. And so our result is as above, which is as we expect actually. Take note that is the position operator, and that **d** is just the vector displacement. Let’s check out the effect of the translation operator on the momentum operator,



We don’t need the expansion because Tcommutes with **p**. Physically this says that if we translate a something with momentum **p**, the momentum remains the same. And this is what we should expect. Let’s consider its effects on the orbital angular momentum operator.



And so as above, which is no surprise. Note the use of inserting 1 in the form of TT† in order to simplify the calculations. Of course we also have:



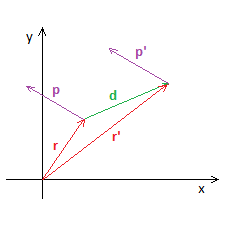
All of these results, you will observe, are intuitive based simply on classical grounds. 90% of the results following you can intuit just be thinking about what it means classically.

**Geometric Interpretation**

So it’s worthwhile to consider this pictorally. Basically the translation operator



does this to the vectors:



So note that each of the individual components transforms as the components of a vector,



**Example**

Explicitly verify that to first order in d.



To first order we can expand the exponentials and write:



so there we go.

**Example**

Consider the following Hamiltonian,



Does this Hamiltonian conserve momentum? Yes because it is translationally invariant. To verify we perform,



Of course it might have been easier to check if H commuted with the **p**-operator itself (which it does).

**Example**

Consider the following Hamiltonian,



Does this Hamiltonian conserve momentum? No because it isn’t translationally invariant. This is because we have:



So it doesn’t conserve momentum. Generally, no position dependent Hamiltonian will, since spatial dependence gives rise to forces, which change momentum.