**Many-Body Correlations**

Will do a quick example.

**Single and Double particle distribution function for Fermions**

Say we have a system of N Fermions occupying the single particle states: |α>, |β>, …., |ζ> (N total). Then:



where ψσα(r) = <rσ|α>. And this makes the density function,



Note that the wavefunctions are normalized so that:



Typically, a state i would be defined to be spin up or spin down exclusively, and so one or the other of its spin components will be zero, however. Okay, so what are the single and double particle distribution functions? Well, fσ(r), which I’m going to denote as fσ1(r1), is given by:



These single particle states are orthonormal, so we’ll get zero basically, unless the permutations match ε12…N = ε1´2´…N´. Then we have:



And each of these guys is orthonormalized so that integrating over position and summing over spin gives 1. So every integral/spin sum will go away for every permutation. That leaves us with just the |ψσ1α(r1)|2 term permuted among the different states α,β,…,ζ. There will be (N-1)! permutations among the r2σ2,…,rNσN terms for every one of the r1σ1 guys. So this will come to:



where the sum over j runs over all allowed states. So we have:



Now let’s consider the two-particle correlation function.



Unless the permutations on the r3σ3 →rNσN terms match, we will get zero. So we can say,



but we must keep in mind that **r**1, **r**2, **r**3, …, **r**N will still be permuted across all the states, and so it’s not just the α and β pair of states that will be outside the integral; rather, every pair will have its moment outside. For every pair that comes outside, there will be (N-2)! arrangements of the pairs inside the integral. So I think we can write this as:



where in the sum, i and j each run over all states, and we’re summing over the permutations of the sets {12} and {1´2´}. Well, I think I overcounted. So summing over i and j separately means that I’ll be counting state where i = α, j = β, say, and also i = β, j = α. But we should only be counting distinct permutations. So have to divide by 2 I think. Now using ε12 = 1, and ε21 = -1, we have:



We can write this as:



where the sum runs over all states i ∈ (α,β,…,ζ) and j ∈ (α,β,…,ζ). What about the single particle correlation function? We have:



Unless the permutations on the r2σ2 →rNσN terms match, we will get zero. So we can say,



but we must keep in mind that **r**1, **r**2, **r**3, …, **r**N will still be permuted across all the states, and so it’s not just the α state that will be outside the integral; rather, every state will have its moment outside. For every state that comes outside, there will be (N-1)! pairs inside the integral. So I think we can write this as:



where in the sum, i runs over all states. So we have:



Now let’s look at averages of single and two-particle energy terms. Say we have a Hamiltonian,



Borrowing our results from the previous file,



where, as can see in previous file, f(r1|r´1) = Σσfσ1|σ1(r1,r´1), f(r) = Σσfσ(r), and f(r1,r2) = Σσ1,σ2fσ1,σ2(r1,r2). We can say,



Let’s decompose the index i into a spatial, p, and spin, s, part. And let’s also suppose that every state is either spin up or spin down, i.e., s = ↑, or ↓, and that the spatial part of the state is the same either way. Then we can say,



So we have:



Now the single particle potential is given by:



And end up with:



Last, we have:



working out the sum over spins,



which comes to:



We can switch p1s1 → p2s2 and vice versa on the last to terms, since these are just dummy indices. Then the top and bottom terms are the same, and the two middle terms are the same. And we have:



So there we go.

**Example: Application to fermion gas (free)**

Let’s apply the preceding to the free fermion gas. Let’s say we’re calculating ground state properties of our gas and we want to evaluate the two-particle probability distribution function.

Well, let’s say we want the identical spin distribution function:



In the last we can switch i and j to duplicate the cross-term. Our basis states are plane waves:



So,



Now we’ll note that ψσks(r) = <rσ|ks>, etc. And this will be non-zero only when s = σ. So the sum over s, s´ will collapse. So then we’ll have:



We’ll work this out later. Now let’s do the unequal spin case.



In the last we can switch i and j to duplicate the cross-term. Our basis states are plane waves:



So,



Now the first term in the summand will collapse the spin sum s, s´, because the first wavefunction guy will require s = σ1 and s´ = σ2 to be nonzero. Then the four-ψ term vanishes because |ks> cannot be both spin up and down, at least for the GS situation we’re considering. ‘cause note that ψσ1ks\*(r1) = <ks|r1σ1> and ψσ2ks(r2) = <r2σ2|ks>, and yeah, we cannot have any state |ks> in the GS have a simultaneous projection onto both spin and down states. That isn’t to say that this could happen for some other situation where our |k> states are conceived to be a superposition of up/down. So we’re just left with:



I’ll defer working that out – though it’s just one more line – because I’m going to repeat everything next. Let’s do everything from the 2nd quantization formula.



Let’s suppose we know the occupation numbers of the ground state in energy space. So then we’ll transform the position annihilation/creation operators to energy annihilation/creation operators (ψ now represents the wavefunctions = planewave·spin and j runs over )



Let’s say we want fσσ. Then,



Note j can’t equal k because if so, then we can see we’d just get zero from the first line.

Continuing, using fact that c’s anticommute, and number operators commute with c’s regardless.



Now we just have the number operator acting on the GS. And this will be zero or one. So in effect the sum will only run over occupied levels,



This is the same as what Mahan has. In the homogeneous gas for example,



this will work out to (switching j→k, k→k´):



(factor of N/2 because only half of particles have spin σ in the GS). Note you can change k´ → -k´ to make it the same as that one behind it. Let’s call it Λ. Then,



So then, going to the continuum vis a vis Λ…



Guess we’ll keep going,



Well that did take forever. Turns out the thing in brackets is a spherical bessel function, -j1(kFr) So we could write,



And therefore,



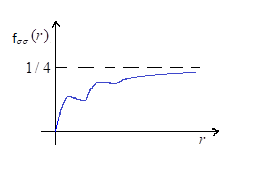
Might use fact that kF3 = 3π2n0 (see Condensed Matter file maybe),



Is the limit of j1(r)/r = 1/3?



Good. So our function, sans the 1/V2 looks like this:



The Pauli-exclusion principle is clearly operative, preventing identical spins from lying on top of each other. If the charges are in a positive jelly background, then this indicates there will be a net positive charge surrounding each electron, because of the dearth of electrons in its vicinity. Now let’s consider the two-particle probability density function for opposite spin particles:



Again, we can separate this into two distinct products,



This time, j *can* equal k. And we can do the commutey/anticommutey stuff as before.



The first one is just the number operator acting on both sides. The second is different. b/c of ckσ, only way we don’t get zero is if kσ is occupied. But then kσ´ is also occupied in the GS. And then ckσ´† will kill it. So it seems we must simply get zero here. So then we have (switching j→k, k→k´):



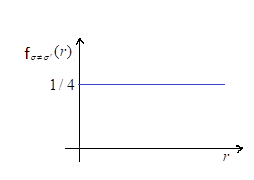
Now our states are:



this will work out to:



So this is constant, and what we’d expect,



that unlike spins are uncorrelated.