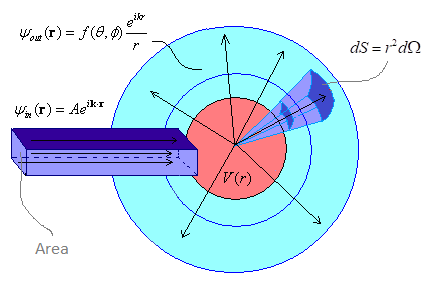
**Scattering in 3D**

Analyzing scattering experiments in 3D is basically the same as analyzing them in 1D. Physically, we set up a beam of particles, with particle density |A|2, traveling to the right (in direction **k**), with energy E = ћ2k2/2m, characterized by ψin(**r**) (this is the rectangular looking thing to the left), and allow them to crash into the potential (the pink sphere). When they hit the potential, they scatter in all directions forming, in the large r limit, a spherical wave, ψout(**r**) (this is the blue concentric spheres (actually this ψout asymptotic form technically presumes the potential drops off at least exponentially with r, but we can sometimes get by w/o this assumption), with energy E = ћ2k2/2m.



In 1D we are interested in the fraction of incident particles that penetrate the potential through to the other side (this is the transmission coefficient T), and the complementary fraction of incident particles that reflect backwards (this is the reflection coefficient R = 1-T). In 3D, we can similarly make a distinction between reflected and transmitted particles. The transmitted ones would be the guys that hit the detector along the same trajectory they were following, basically at the angle Ω = 0 (defining 3D angles from the incident beam’s direction, ). This is also called forward scattering. The reflected beam has many options, though; particles can be reflected back along any direction ´ (not equal to ), which I’ll synonymously refer to as the 3D angle Ω (not equal to 0). Given any incident beam cross-section area, which in the diagram I’ve simply denoted as ‘Area’, some fraction of the area will transmit, and some fraction will scatter/reflect. The cross-section area of the beam which *scatters*, i.e., doesn’t *forward* scatter, is called the scattering cross-section area, σ, of the beam. And then the area that transmits would be: Area – σ. For tiny beams, σ is presumably equal to something just below Area (must be at least a *little* less than Area b/c there is a quantum mechanical probability that particles within any beam size can forward scatter, even if it would be classically forbidden). But as Area approaches ∞, I think the scattering cross-section area will approach an asymptotic limit, σT. This target-dependent maximimum scattered area is called the total scattering cross-section σT. The total scattering cross-section area can be thought of as the cross-section area within which the target’s force is active. Classically, for a contact force, the total scattering cross-section will be σT = πR2, where R is the radius of the target ball. But thanks to QM, σT will typically be a little bit more than this, like 4πR2 I think (actually a little more than *this*). If the force is long-ranged, but exponentially damped, like Fexp(-r/μ), then the total scattering cross-section is still something like σT ~ 4πμ2 I think. But if the force goes as a power law, like F(μ/r)n, then the total scattering cross-section will truly be infinite. But while σT may be infinite sometimes, we should observe that σ itself will never be. Rather we must always have σ < Area. So σ is a property of the beam, but σT a property of the scatterer. So having said this, we can crudely think of the transmission coefficient in 3D as being something like T = (Area – σ)/Area, and reflection coefficient as being something like R = 1 – T = σ/Area. For a very large beam Area >> R2 say, then this is approximately T = (Area – σT)/Area, and R = σT/Area. In these notes at least, we will always be dealing with an approximately infinite cross section incident wave, and so we’d be dealing with only the latter set of definitions. But for convenience sake, we’ll still refer to σT as σ, in that context.

We can examine the probability distribution of the *scattered* beam, P(Ω≠0,σ). We don’t concern ourselves with the *unscattered* particles, if any, as their behavior is known, and in any event, it is impossible to include them in the probability distribution P(Ω,σ) because as Area grows way beyond R2, the additional particles will have a greater and greater likelihood to simply *forward* scatter, and so we’ll mainly just be changing the distribution at the single Ω = 0 point…but a p.d.f. isn’t responsive to change at a *single* point. So suppose we have a given incident scattering cross section, σ, with incoming current Iinc., and we want to know what fraction of this current (i.e., the scattered current) will scatter through the angle Ω, within range dΩ. This would be what?



where we define the ‘differential’ scattering cross-section:



and in this context, interpret the dσ as the amount of scattering cross-section area of the beam, σ, that get’s scattered into the solid angle dΩ = sinθdθdφ. And so we have the probability distribution can be obtained simply by dividing dσ/dΩ by σ itself. We’ll also observe that we can get the scattering cross section area back by calculating σ = ∫(dσ/dΩ)dΩ.

A caveat on using infinite Area incident beams. For finite range potentials, really, for potentials with a finite value of σT, the value of σ and the expression for dσ/dΩ approaches a limiting value for Area >> R2 (for the most part). And so it is easiest to not bother about the area of the beam, and just presume it to be infinite. But for infinite range potentials, like the Coulomb potential, for which σT is itself infinite, we will get non-sensical answers like σ = ∞ (naturally), and dσ/dΩ = ∞ (as approach Ω = 0). We can ad hoc cure this by imposing a small Ω cutoff, perhaps estimated from classical physics, on dσ/dΩ (see Classical Mechanics file).

**Formula for the differential scattering cross-section**

Now we want to work out the formula for the differential scattering cross section. So,



and,



And so,Ω



Note that our analysis above implicitly presumes infinite incoming cross-section area, Area. Area = ∞ should be a decent approximation for any finite beam for which Area >> σT. An infinite Area means that we should always get ∫dΩ(dσ/dΩ) ≡ σ = σT. But this will obviously run us into problems for cases where σT = ∞, such as with the Coulomb force. So how to reverse engineer the result for a finite Area beam? As discussed in the time-dependent scattering file, I believe this will simply mean that our differential scattering cross-section fails for the small θ / near-forward scattering, regime. The classical resolution would be to simply impose a small angle cutoff θmin corresponding the classical trajectory of a particle in the circumference of the beam. Or maybe one should impose a cut-off such that ∫(dσ/dΩ)dΩ = Area. But strictly speaking, there should be no abrupt cut-off since there is always a finite probability to scatter into any angle for any finite beam Area.

**General methodology for determining the scattering cross-section**

To get dσ/dΩ, we use the same basic procedure that we used in 1D scattering. We solve the Schrodinger equation for the potential V(**r**), and for particles with energy E = ћ2k2/2m (where k is a known parameter, unlike our usual case where E is an unknown we’re solving for).



Then we take the large r limit of ψ(**r**) to extract the form,



and then identify the scattering cross-section as,



Extracting the asymptotic limit from the general solution of the Schrodinger equation, ψ(**r**), is a problem itself, which we will consider now.

**Solving the Schrodinger equation in the asymptotic large r limit**

Suppose, first of all, that we have a spherically symmetric potential so that V(**r**) = V(r). The Schrodinger equation is then,



We can assume a form,



which yields, filling in the radial derivative, and dividing out the constants,



We want the asymptotic limit of the wavefunction, and for short ranged potentials the V(r)r2 term will go to zero in the large r limit. This would be the case for δ function potential certainly, the spherical box potentials, Yukawa potential (discussed below), etc. But this assumption about V(r)r2 → 0 doesn’t apply to the Coulomb potential, which would have to be handled with more care. Making this assumption in any case, our Schrodinger equation comes to:



and the solutions, which you may verify from our past discussion of spherical potentials is just a linear combination of spherical Bessel and Neuman functions,



What exactly B and C are would be determined by whatever relevant boundary conditions the wavefunction must obey, etc. Taking the large r limit of the radial function we would have:



So asymptotically, the general solution to the Schrodinger equation, with energy E = ћ2k2/2m, will look like,



**Matching up the asymptotic SE solution to the scattering form**

What we have to do now is take this solution and somehow extract from it the asymptotic form



First off, we’ll start with a little simplification. We do not expect f(θ,φ) to be a function of φ, due to spherical symmetry, and so we may set cℓm = cℓδm0, which turns the spherical harmonics into Legendre polynomials. So now we have:



Going to stop right here for a moment and observe that our form shows clearly we have spherical waves coming in (the left term), and spherical waves coming out (the right term), which are modulated by a phase shift. This is in good analogy to the 1D Parity conservation case, where incoming ± parity waves were bumped into outgoing ± parity waves, with an accompaning phase shift, by the potential. Continuing,



The first term is going to be the ei**k**·**r** guy, and the second the f(θ,φ)eikr/r term. To see how, we must use the identity,



where the Pℓ(cosθ­) are Legendre polynomials. For instance, starting with ℓ = 0:



The veracity of the formula for higher ℓ’s can be proved using the recursion relation for j’s and P’s. In any event, using this formula, and the orthogonality conditions for Legendre Polynomials,



(the last line is nothing other the statement,



except for the fact that we have to explicitly divide by the normalization 2/(2ℓ+1) in order to normalize the Legendre Polynomials). So we have:



So we have:



And we see that we would need to make cℓ = Aiℓ(2ℓ+1) in order to extract the term Aei**k**·**r**. Doing that we get:



And now we see that:



and so the cross-section formula dσ/dΩ = |f(θ)/A|2 is, simplifying a little:



So we see that the scattering cross-section is determined entirely from the phase shift, δℓk, which itself is determined solely from the radial part of the Schrodinger equation. So all we have to do is solve the radial part of the SE, get the phase shift δℓk, and plug it into the formula above and we’re set. The scattering cross section, σ = σT, is:



We define the partial cross-section and say:



In general, the scattering cross-section seems to depend on ℓ and k. It seems we can interpret σℓ/σ as the fraction of scattered particles that emerge with momentum ℓ (well I guess none of them must emerge with any particular ℓ, but can exist in a super-position, but if we do *measure* ℓ, and thereby force an ℓ value on them, then we’ll get that fraction).

Should also mention that σ in general, and σℓ as well, are kind of analogous to the reflection coefficient, R, in 1D scattering. If σ were ever 0, then that would mean the beam just forward scatters, i.e., doesn’t scatter. I think σ = 0 is unlikely to happen, but we can definitely get σℓ = 0.