**Time Development in Schrodinger Picture**

There are two equivalent ‘pictures’ regarding time-development, in quantum mechanics. One is the Schrodinger picture, in which the state vector is considered to evolve with time. The other is the Heisenberg picture, in which the operators are rather considered to evolve with time. They are equivalent, basically…

So first we’ll consider the Schrodinger picture. We can motivate our time-development postulate by using an analogy with the old quantum theory. In the old quantum mechanics, the wavefunction looked like,



Taking the time derivative, we get the equation:



And so it seems natural to suppose that the state vector evolves in time according to the equation:



where H is the Hamiltonian. This is the most important equation of the semester. The Schrodinger equation is a more general statement about the time evolution of |ψ> than is the original statement that ψ(**r**,t) = ψ0ei**p**·**r**/ћ-iEt/ћ because the solutions to the Schrodinger equation don’t necessarily look like ψ(**r**,t) = ψ0ei**p**·**r**/ћ-iEt/ћ.

Now remember from the last lecture that when we project an operator equation onto a certain basis, it becomes a differential equation. Let’s project this operator equation onto the position space basis and see what we get:



and so we get the celebrated Schrodinger (partial differential) equation (in position space).



Note that there are other bases besides, |**r**>, which we can use to project the Schrodinger equation (the operator form) onto. And we will do so at various times. But this is the one most often used for sure. Note that if there is some time-dependent force acting on our particle, F(r,t), then V would be upgraded to V(r,t) such that F = -∇V. But in that case, H cannot be considered the ‘energy’ of the particle per se´, just the Hamiltonian, cause there is no potential energy associated with a time-dependent force.

**Solving Schrodinger Equation for time-independent H’s**

Now suppose we have a particle in the state ψ0(x) at time t = 0. What will the state of the particle look like, at some time t later? To determine this we will need to solve the Schrodinger equation with the initial condition |ψ(t=0)> = |ψ0>, i.e., we need to solve:



**Partial differential equation method**

There are two ways to do this. One way is to put this operator equation in position space, at which point it becomes a partial differential equation, which can be solved by the appropriate methods. And another is to solve the operator equation and then put it in position space. Let’s try the former first. So inserting the Ĥ operator, and taking the projection against <x| we get:



and the initial condition goes to:



So we get the equation and initial condition:



When solving any linear partial differential equation, the usual approach to try is separation of variables. So we will assume that ψ(x,t) can be written as ψ(x,t)=T(t)X(x), and plug this into the differential equation.



Now the only way that two functions of completely different variables can be equal is if both are equal to constants – should think about this to make sure you agree. If so then we can say:



which means,



where the constant E is technically just some arbitrary constant, but we can identify it as the energy because you will notice that the X equation is just the Energy eigenvalue equation. This is always the case. OK so now we have reduced the partial differential equation to two ordinary differential equations, which we can now solve. The solutions are:



The solution to the time equation is:



Now suppose the solutions to the Energy eigenvalue equation are:



Then the solution of constant E is:



But really, E can be any energy eigenvalue. So our most general solution is:



Almost there. Now to answer our question about how a state initially in ψ0(x) will evolve. What we want is to determine the cE’s so that:



To determine what these cE’s are, it helps to write our equation as this:



Now remember that the |ψE> eigenvectors are assumed to have been orthonormalized. So we can say:



Putting this expression in the position basis, we have generally that:



So the general solution to the Schrodinger equation can be written,



**Operator method**

So that’s great. But, the structure of quantum mechanics usually lends itself to a more operator-centric method of doing things. So let’s revisit the solution of the equation from that perspective. The solution comes out faster this way, and it is usually how this is done in quantum mechanics, except in rare circumstances where V may depend on x and t in an inseparable way which makes this less useful. Alright the idea is this. Go back to the original Schrodinger equation with initial condition |ψ(0)> = |ψ0>



Suppose we want to solve for the vector |ψ(t)>, with the initial condition that |ψ(0)> = |ψ0>. Let U(t) be the time-development operator that evolves |ψ0> in time from |ψ0> to |ψ(t)>, i.e.,



Now we want to determine an equation for the operator U(t). So plug it into the Schrodinger equation and initial condition:



Since the operator equation is true regardless of the initial state we are evolving, it must be the case that:



So this is the equation of motion of the time-development operator, U(t). Now we want to solve it. If we were to take liberties and treat H as if it were a scalar, then we would have a simple first order linear differential equation, and the solution would be:



where C is some constant. C must be 1 in order to satisfy the initial condition. And this turns out to be the correct result:



So the operator solution to the Schrodinger equation is just:



Using this fact, we can quickly determine how the wavefunction evolves in position space. We first apply the time-development operator to |ψ0>. Then we insert two Î operators – one in the energy eigenbasis to enable evaluation of Ĥ, and another in the x-basis to put the |ψE>’s in the position basis – which is the basis we usually know them in.



So the solution is:



which is precisely what we determined previously.

**Few quick properties of U(t)**

First note that



So U(0) is just the identity operator, as it should be since at time t = 0, |ψ> hasn’t yet evolved away from |ψ0> and so U(0) should just return |ψ0> back. Secondly, define the time evolution operator from time ta to tb, written below.



Then this obeys the composition property, which we’ll use later:



This states that the evolution of the particle from t1 to t3 is just its evolution from t1 to t2 × its evolution from t2 to t3, which makes intuitive sense perhaps. But you can explicitly verify that this property is obeyed by U(tb,ta).

As a matter of notation, U(t) would really be written U(t,0), the time evolution operator from 0 to t. But the zero is usually left out for convenience sake.

**Harmonic Wavefunctions**

I’ll pause to mention that eigenfunctions of H develop harmonically in time, according to:



and so, if one happens to find a wavefunction which develops harmonically, then one can immediately identify its energy as En = ℏωn. And as we see above, even more general wavefunctions reveal the energy levels of the system, since any arbitrary wavefunction develops in time according to:



If we were to say, take the Fourier transform of this function with respect to time, then we’d have:



and so the poles of this construct would be precisely the energy levels of the system. This is useful to know because sometimes it is easier to work out the time-development of a system, and obtain its energies indirectly as alluded to above, than it is to try to compute the energies directly.

**Solving Schrodinger Equation for time-dependent H’s**

Now let’s say H is time-dependent, and we need to solve something like:



**Partial differential equation method**

We can try the partial differential equation method again by projecting our equation onto position space…and we’ll get:



This time we cannot use separation of variables. And so we’d have to use some other methodology, specific to the PDE. Other option is the operator method again,

**Operator Method**

Above our U(t) operator was easy to work out because H didn’t depend on time itself. But what if that’s not the case? Then consider the general case:



and we can write this as:



again, this equation is true regardless of the initial state, |ψ0>. So it is true that:



So what is the solution to this equation? Unlike before, we cannot naively treat H(t) as a constant and solve it like a scalar first order differential equation. The reason for that will become evident shortly. So what we do instead is this. We solve the equation perturbatively, treating H(t) as a perturbation. So stick a λ on it, and expand our time-evolution operator in a power series in λ.



(I’m putting the former superscript (n) in the subscript so it doesn’t get in the way of other things). Plugging it in we get:



Equating power by power:



The first equation is easy to solve. It’s just a constant. And imposing the initial conditions, we have:



The second equation can be easily solved as well, along with the initial condition that U1(0) = 0, so we don’t mess up having satisfied the initial condition with the previous term.



Alright. What’s the second term? Doing similarly, we have:



Similarly, the third order term obeys the equation,



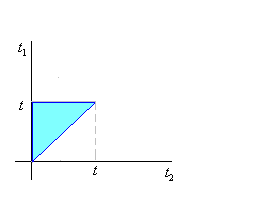
So we see that we can write our result as:



Now let’s look at the second term in the brackets.



The integration region runs over the following triangle in (t1,t2) space



But the integrand is symmetric since if we switch t1 and t2 we get the same operators back – almost. Actually switching t1 and t2 doesn’t quite give us the same thing back because the operators H(t1) and H(t2) usually don’t commute.



But the integrand will be symmetric if we just agree to keep the ordering of the operators the same. Note how t1 > t2, and how the order of the operators is H(t1)H(t2) so that the ‘later’ time occurs on the left. If we agree to always keep the operators in that order ‘later times to the left’, then the integrand will be symmetric about the diagonal and we can extend the region of integration to the lower triangle (and divide by 2 to compensate). So then we can say:



where T is the ‘time ordering operator’ which orders the operators according to the latest time arguments to the left. For example,



Or formally,



Similarly, we can write the third order term as:



and generally,



and with this we may write our result as:



where the last expression is to be interpreted as signifying the previous line. So altogether we can say that our time-evolution operator looks like (setting λ = 1)



We cannot remove the T symbol unless H(t) commutes with itself at different times, and so that is why our intuitive answer U(t) = exp(-i∫0tH(t´)dt´) wasn’t quite right.