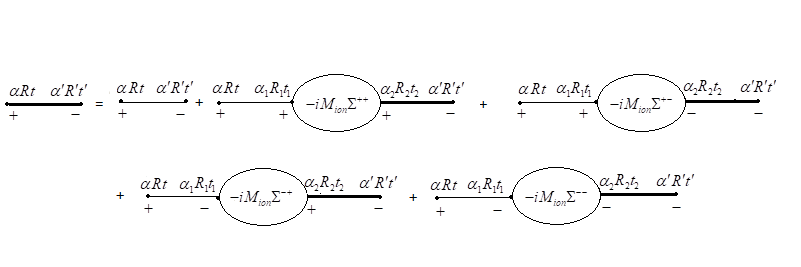
**Self-Energy**

**Self-Energy for GR (possibly time-dependent interactions)**

It seems plausible that the recursive equation for GR, and its concommitant self-energy, can be formulated as is done in the identical multi-particle case. And that everything would carry over straightforwardly. Presuming so, we can examine the self energy concept in the context of the general contour ordered GF, which may include time-dependent interactions. The self energy contains the energy corrections, lifetime corrections, and spectral weight of the excitations in question. It is the same as the information obtained from doing the TDPT in QM. So consider the self-energy expansion for G+-, for instance. We have:



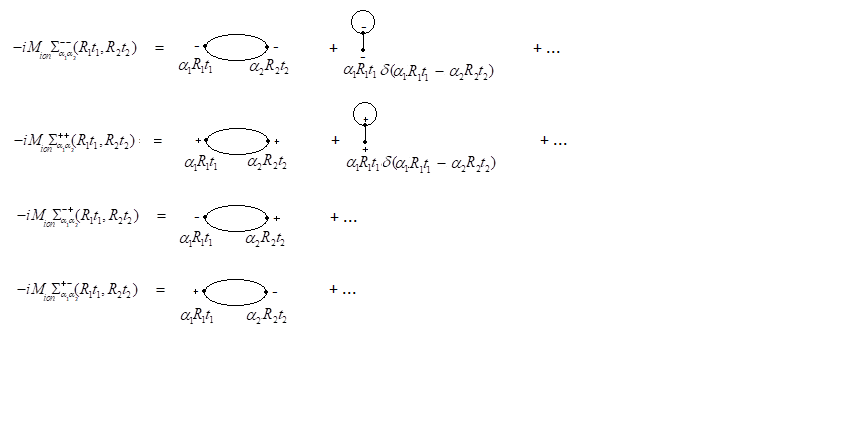
This equation would read (defining the bubble as -iMionΣ, whereas its usually defined as just -iΣ, but I like the form it comes out with better so….):



which simplifies a little to:



Some of the self-energy terms, for the *cubic* interaction would be:



(note the δ(α1R1t1 – α2R2t2) is three separate δ’s really and the ones for the α’s, R’s are Kronecker deltas). We could include external field interactions as well, if desired. And similarly for the other 3 GF. As in the single particle file, we can write these four equations as a matrix equation:



with



Additionally, we know that the free GF’s satisfy the differential equation,



(can show this to be true if look at work from Formal Properties file) and if we apply this fact to the equation above, then we’d have,



Despite the symmetry of the formulation, it doesn’t take into account the fact that the GF’s are linearly related. To reduce the equations to the only degrees of freedom, we can make the following matrix transformation on the equations. We define:



Then the transformed matrix, G´, is:



Now from the relations in the Formal properties file:



this simplifies to:



where we define F = G-- + G++. The transformed self energies are, Σ´:



To make progress on this, we can use the following relationship:



which will eliminate the lower right matrix element, as can see. Before proceeding, let’s prove that relationship. So we’ll start from which follows from: GC + GAC = G> + G<. Then if we insert the self energy equations GC,AC,>,< into this relationship (leaving summations over indices and position/time integrals implicit) we have:



and then apply the differential operator –(Mion∂t2 + Σα´´R´´Kαα´´(R-R´´)), which I’m just going to write as -(M­ion∂t2 + ) to save space, to each side of the equation. And we’ll get, using some notational license:



But now recall from formal properties that:



which implies,



and so the last line simplifies to:



and so we have,



which is what we wanted to prove. Okay now let’s go back to the self-energy equation and use this relationship. We’ll have:



where we define Ω = Σ-- + Σ++, ΣR = Σ-- + Σ-+, and ΣA = Σ-- + Σ+-. The usefulness of this formulation lies in the fact that some of the equation decouple. Consider the recursive matrix GF equation again, leaving the summations, integrals implicit.



Inserting our R’s, we have:



And inserting our results for these guys in, leaving integrals implicit, again,



The nice thing about this is that the equations for GA and GR are entirely self-consistent, excepting the fact that their respective self-energies involve other GF’s anyway. So explicitly we have:



Therefore, we *can* write out a self-consistent equation for GA, or GR in terms of just itself. If we operate on both sides of the retarded one, say, with –(Mion∂t2 + Σα´´R´´Kαα´´(R-R´´)),



which gives us an integro-differential equation from which one might attempt a non-perturbative calculation of GR. The nice thing about this equation is that it clearly decouples GR from the rest of it, something the exact differential equation in the previous file did not do. Of course, this equation would only be as accurate as the exact coupled one in the limit that we include all terms in Σ. You might think of the integral term as a sort of self consistent single particle potential approximation to the interaction. One could attempt to solve this equation iteratively.

For time-independent, homogeneous systems (this would mean no single particle potential, and constant λ coupling, but doesn’t have to be isotropic don’t think), the GF’s and Σ’s would be functions of just the difference of the time and spatial arguments. Then starting from the highlighted equation (for GR say) we can go to Fourier space (see Fourier transform file perhaps) and obtain,



Inversion of this equation yields,



where 1/ and -1 in this context means ‘matrix inverse’. For simplicity, let’s further specialize now to an isotropic medium so that K is diagonal in its indices. Then G0 and Σ are diagonal too. Checking out the non-interacting GF file, we see we have:



which makes GR come to:



where we’re denoting Σαα´R(**k**,ω) = δαα´ΣR(**k**,ω). Given this form, we can obtain the spectral function:



The poles of the Green’s function determine the excitations, and so we will want to expand the Green’s function about its pole.



Suppose we find the root of this equation ω2 = [we’ll impicitly note that Σ is a function of ω2 because ω only appears in the G’s as squared]. Then we’ll expand the denominator of the Green’s function,



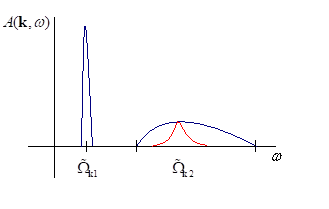
which we can write as:



Zk is called the renormalization factor. Z < 1 of course. The closer it is to 1, the closer the approximation is to a real eigenstate. Would the root **k** just be the frequency of oscillation about the new minimum? I guess not entirely because there could be an imaginary part as well. The spectral function will come to:



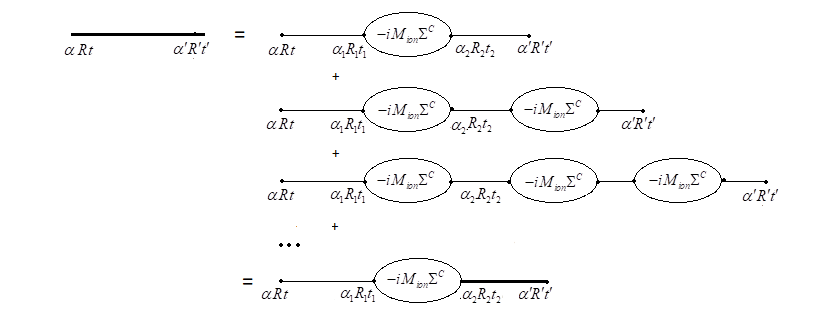
and again, the identifications make sense. Note how ImΣR must be less than 0 for A to be positive. A typical spectral function, A(**k**,ω), will look like below (this would be before we made the approximation about a single pole that we just did), in blue.



The presence of two peaks indicates that there are two excitations of the system – could be that our potential is of such a form that it has two minima, instead of the single one our unperturbed system has. The width of the peak is given by the ImΣ. So the delta function thing on the left would have that ImΣ = 0, while not so for the right hand curve. If we tried to approximate the latter with a Lorentzian, as we do above, we’d have the red curve, whose location gives the energy best associated with it, and whose width would give the scattering rate basically, and whose height would roughly give the renormalization factor, basically the amount that this would resemble a quasi-particle.

**Self-Energy for GC (time-independent)**

The above applies still, but if things are time-dependent, then we have a self-consistent equation for GC as we saw in previous file. So I’ll formulate everything in terms of it. We can sum a subset of diagrams exactly. For instance,



which is equivalent to the equation:



The difference between this and what we have above with GR is that ΣC, in this case, consists entirely of G0C’s, whereas above ΣR contained all types of the GF0’s. Let’s say that λ is constant, and we have no external potential. Then the system would seem to be homogeneous, and Σ would depend only on the difference R1 – R2, and t1 – t2. This would allow a spatio-temporal Fourier transform on our recursive equation. And we’d have:



We can solve this equation most easily by treating it as a matrix equation in polarization space. Let G, G0, and Σ, w/o indices, represent the respective matrices with elements Gαα´, G0(αα´), and Σαα´. Then our equation reads, and can be solved….



where -1 means matrix inversion. We usually write this more cleanly as,



where, like we said above, 1/ and -1 means matrix inversion in this context. Now let’s say we have an isotropic system so that K is diagonal in its indices and so G0 and Σ are too. Filling in the form of G0-1 and denoting ΣCαα´ = δαα´ΣC, we have:



The poles of the Green’s function determine the excitations, and so we will want to expand the Green’s function about its pole, ω = k, defined by:



Suppose we find the root of this equation ω2 = [we’ll impicitly note that Σ is a function of ω2 because ω only appears in the G’s as squared]. Then we’ll expand the denominator of the Green’s function,



which we can write as:

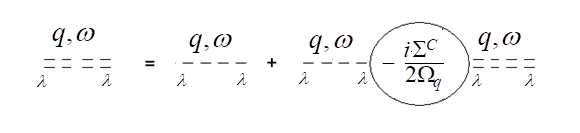


Z is called the renormalization factor. Z < 1 of course. Would the root k just be the frequency of oscillation of the kth mode? I guess not entirely because there could be an imaginary part as well. Anyway, this allows us to identify the self-energy term as a correction term including the effects of the external potential. To the best degree possible it provides a description of the energy of GS as being comprised of harmonic oscillations with wave-vectors, **k**, and oscillation frequencies k.

It is worthwhile to see the self-energy structure of the DλC(**k**,ω) Green’s function too. Recall that this was defined via:



The self-energy equation for D would look like this (now I’m representing the exact D with double dotted lines, and I’m propitiously guessing what the self-energy bubble is, in terms of Σ, as we’ll see below). I’m assuming the interaction, whatever it is, is homogeneous / isotropic / whatever so that different polarizations aren’t mixed (except within ‘Fermion loops’ where they would be summed over).



The recursion relation gives us:



which is:



which is, finally:



and if we plug this back into the equation relating G and D, we see we get our previous G result back,



The sum over λ just turns the vector outer product into δαβ. So this matches our GαβC(**k**,ω) result.