**General Statements**

I just want to consider some general principles that apply even when we can’t figure out the time-development of a particle. This will basically just be a repeat of one of the files in the Foundations folder.

**Energy-time uncertainty relation**

Going back to Ehrenfest’s theorem,



We’ll note that if Â commutes with Ĥ then the expectation of A is constant with respect to time – provided Â itself doesn’t explicitly depend on time. A corollary is that the expectation of the energy is always constant with time (a statement of the principle of conservation of energy) since Ĥ commutes with itself. Another statement we can make is that if the state vector is itself an eigenvector of the Hamiltonian, then the expectation of A will be constant, regardless of whether Â commutes with Ĥ or not. This is because,



Another point of interest is that if A commutes with H, and |ψ> starts off in an eigenstate |ψA> of A, with eigenvalue *a*, then it will remain in a state with that expectation.



Now as we argued in that Commuting operators file, if [A,B] = 0, then [A,f(B)] = 0. And so since [A,H] = 0, we have [A,U] = 0. So we can say,



So that shows us |ψA(t)> will remain an eigenstate of the operator , with eigenvalue *a*. But do note that |ψA(t)> can be *any* eigenstate with that eigenvalue. So recapitulating:



where all the expectations are understood to be with respect to the state |ψ>. OK, now suppose though that Â doesn’t commute with Ĥ, and that |ψ> is not an eigenstate of Ĥ. And we’ll suppose that |ψ> isn’t an eigenstate of A. Then presently |ψ> will have some average value <A>, and <E>, as well as some uncertainty ΔA and ΔE. And since |ψ> isn’t an eigenstate of H, these expectations will change with time. Let’s figure out how long it will take for the expectation of A to pass through one present standard deviation from the mean, ΔA. According to the Heisenberg uncertainty principle we have:



Assuming Â doesn’t explicitly depend on time, Ehrenfest’s principle will enable us to say:



Let Δt be the time it takes <Â> to pass through one standard deviation. This would be:



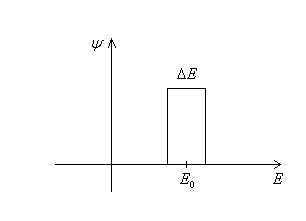
and so then we can write the relationship above as:



This expression tells us that it is the uncertainty in the energy that controls how quickly the state decays from its initial configuration. If the ΔE = 0, i.e., the state is an eigenstate of Ĥ, then all observable’s expectations remain constant. But if the state isn’t an eigenstate of Ĥ, then all observable’s expectations will change, and the larger the uncertainty in E, the quicker it will change. But now we shouldn’t interpret this decay as necessarily decoherence, it’s just some unspecified change of form.

**Energy-time uncertainty relation**

Suppose we have an energy wavefunction ψ(E) with energy uncertainty ΔE. Say something like a box of width ΔE centered about E­0.



What is its lifetime? For this we must calculate ψ(t). Well,



So we have,



and we see that this is a decaying (oscillating) function of t. Now what do we mean by ‘lifetime’ of the state? We would mean the amount of time it takes for the wavefunction to lose its character of being peaked, box-like, in energy space. Note that at t = 0, the coefficient of exp(-iE0t) is 1. As t progresses, it will become smaller. Eventually it will go to 0 (and then back up again, and back down, etc.)



We can say that it loses its character by the time it first touches down at 0. This would be after a time Δt such that



So again, we have here,



**Example**

Let’s look at it more formally. Let’s go back to:



which we’ll just write as:



And let g(E) be highly peaked about some E0. We can do a Gaussian approximation for g(E), this would be:



where E0 is the peak, and ΔE is the spread about that peak. With this supposition, we can do the integral,



From this formula, we can see that ψ(t) will spread out over time, gradually losing its distinctive e-iE\_0t character. The spread, in time, is Δt = 1/ΔE, as can see from the exponent. So again, we have:



**Example**

Suppose we want to model nuclear decay of an alpha particle. So we treat it as a particle in a spherical well of depth V0 = -1MeV, with radius R = 1fm. Let’s suppose it lies in a spherically symmetric state – perhaps the ground state of an infinite well of same size. The generic states of an infinite well are:



where a few j’s are:



And the zeros are:

|  |  |
| --- | --- |
| Spherical Bessel function | Approximate zeroes |
| j0(x) | c01 = π, c02 = 2π, c03 = 3π, c04 = 4π |
| j1(x) | c11 ≈ 4.5, c12 ≈ 7.7, c13 ≈ 10.9, c14 ≈ 14.1 |
| j2(x) | c21 ≈ 5.8, c22 ≈ 9, c23 ≈ 12.3, c24 ≈ 15.5 |

So for ground state, we have:



And j1(π) = 1/π. So our wavefunction is:



Making sure it’s normalized,



So good. Now let’s estimate the energy uncertainty. First,



This is what we would expect. But it’s nice we got it again. Now do,



So now let’s do the kinetic energy squared,



Oh yeah. Of course it is, because ψ0 is an eigenstate of the kinetic energy operator. So then, putting it all together,



Duh. So the variance of the energy is:



So why is it that we have an exact eigenstate, when its not an exact eigenstate? I think its because we’re not accounting for the derivative at the boundary r = R. ψ(r) goes to 0 there linearly. So I think a double derivative would give rise to a delta function there I think. And so there should be a non-zero contribution to the kinetic energy at the boundary. I don’t want to work that out though.

**Cyclic Behavior**

Might note that our time-development formulas suggest periodic behavior. Go back to:



And suppose the eigenvalue spectrum to be discrete, as it should be for any finite volume, no matter how large. And let Egcf be the greatest common factor of all the energy levels. Then each state would have an energy equal to Egcfnstate, where nstate = Estate/Egcf. So we could write:



And so when t = 2πℏ/Egcf, the state will be:



meaning, it has repeated itself. For instance, in 1D, where En = E0n2, and so every T = 2πℏ/E0 the motion will repeat itself.