**HO in Heisenberg Picture**

Going to do two things here: use the time-development of operators (from Heisenberg picture) to obtain the excitations of the HO, and then keep using the Heisenberg picture to calculate the time-development of some expectations.

**HO Hamiltonian**

So we can calculate the time-dependence of expectations of quantities by calculating the time dependence of the wavefunction. But sometimes it’s easier to solve the operator’s equations of motion. Let’s consider this approach. Just to do things differently, we’ll look at the Lagrangian of our harmonic oscillator.



For fun, let’s construct the Hamiltonian of the system, its eigenvalues (even though we already know them) and its eigenfunctions, etc. First,



So



as we know (and from equations of motion we’ll get in a bit, we can know that HH(t) is just HH(0) = H. Basically H commutes with itself at all times because energy is conserved. And that is why it’s also equal to itself at t = 0.

**Illustration that H is constant in time**

Let’s demonstrate this. To do so, we need to know how xH(t) and pH(t) evolve in time. The time-development of the x-operator follows from:



which is just the classical equation of motion as promised. To get an equation for xH all by itself, we can take another derivative.



So we have:



which is precisely the equation for the harmonic oscillator. Another way to get the equations of motion for xH(t) is straight from the Lagrangian.



Of course we may solve it to obtain,



where 0 and 0 are unknown operators satisfying some initial conditions. Assuming we’re looking at the time-development starting from t = 0, then H(0) = as would be clear from the expression:



since (0) = 1. Furthermore, from the first equation we got for H(0) it follows that H(0) = /m. Using these initial conditions we can determine that the unknown operators 0 and 0 to write,



It follows from



that:



Now plug these into HH(t).



So there we go.

**Energy levels of the HO**

We can go back to:



and write it as:



Therefore we can identify these operators as proportional to the creation/annihilation operators (see Foundations folder/Heisenberg Time-development stuff):



Normalizing our ‘creation/annihilation’ operators so that



we have



Of course the last follows from the first directly. And we will indeed recognize these as the creation/annihilation operators we found before. We can go the other way and express xH(t) and pH(t) in terms of these operators, to obtain their so-called ‘free field expansion’.



and using:



we get:



Plugging these into H, will give su:



And clearly now the eigenvalues are indeed ω(n+1/2). Let’s pause to note that we clearly have quantum length, and momentum scales,



These are clearly the ground state amplitude and max momentum since when we plug these values in alone we get the ground state energy.

**Dynamics**

Suppose a particle starts off in the state below, under a harmonic force:



What is its expected position, momentum? How would these expectations develop in time? Well initially it would be:



and as far as the momentum is concerned,



To determine the expectation of x and p as a function of time, we would usually determine the time development of |ψ>, and so we would write,



So we would need to calculate the time-developed |ψ0>. This would be:



and we’d have to then insert a resolution of identity in terms of the harmonic oscillator eigenstates,



Filling this into our formula for x(t) we get:



This formula obviously sucks. That is why the Heisenberg picture is so useful. It suggests that we instead apply the time-development operators to  rather than |ψ0>. So let’s go back to our original expression. We can write it as:



And we know from before, somewhere, that this is equal to:



And now filling in our results for these expectations we get:



Voila! See how much easier that was. This is the power of the Heisenberg picture. And notice how physically intuitive this result is. We start with a classical wavefunction centered at position x0, and when we let it time-develop we see that its expected position obeys exactly the same time-development as classical laws predict.

**Example**

A measurement of the energy of a harmonic oscillator system yields the results ℏ𝜔/2 and 3ℏ𝜔/2 with equal probability. A measurement of the position would have yielded the result ⟨𝑥⟩=−√(ℏ/2𝑚𝜔)sin(𝜔𝑡) . What is the expectation value of the momentum?

The first line tells us that that wavefunction exists in a superposition of the lowest two energy states. The phase can be anything.



The position operator is:



And its expectation is:



Comparing to the stated expectation, we see that: φ0 – φ1 = π/2. Now then, the momentum operator is:



We can find the expectation,



Could’ve also done,



Comparing to: ⟨𝑥⟩=−√(ℏ/2𝑚𝜔)sin(𝜔𝑡), we could get <x> and <p>. And then we could get:



**Example**

What are the energy levels of this H?



One way to do this is to work out the time-development of the creation/annihilation operators. So let’s do,



and,



So altogether,



In matrix form,



Eigenvalues of the matrix are:



These are the excitations. But shouldn’t they be real, for all possible values of ε0 and ε1?